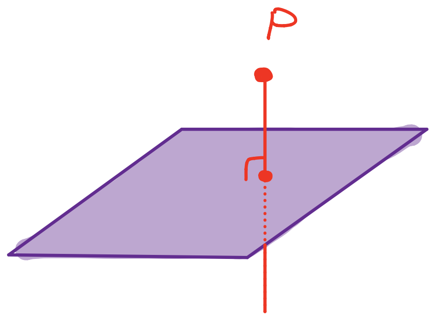
Projections and planes

Question: Suppose we have a point P and a plane in $\mathbb{R}^3$. How do we find the point on the plane that is closest to P?

Answer: Find the line L through P that is perpendicular to the plane, and find the point where L intersects the plane.



In order to do this, it is helpful to use properties of the dot product:

The dot product + angles

Recall that the dot product of two vectors

$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad \text{and} \quad \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad \text{is defined}$$

$$\vec{v} \cdot \vec{w} = x_1 x_2 + y_1 y_2 + z_1 z_2.$$

Properties of the dot product:

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$).

1. $\vec{v} \cdot \vec{w}$ is a real number (i.e. a scalar).

② $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$.

③ $\vec{v} \cdot \vec{0} = 0$

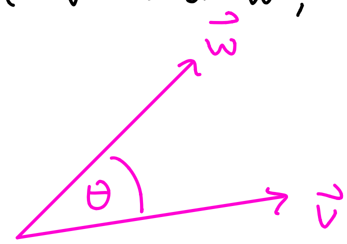
④ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

⑤ If k is a scalar, then $(k\vec{v}) \cdot \vec{w} = k(\vec{v} \cdot \vec{w}) = \vec{v} \cdot (k\vec{w})$.

⑥ $\vec{u} \cdot (\vec{v} + \vec{w}) = (\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{w})$

⑦ If θ is the angle between $\vec{v}$ and $\vec{w}$, then

$$\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$$



We can use property ⑦ to compute the angle between two vectors.

Ex: Let $\vec{u} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

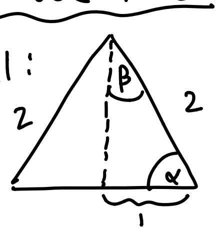
Then $\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$, so

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{-2 + 1 - 2}{(\sqrt{1+1+4})(\sqrt{4+1+1})} = \frac{-3}{6} = \frac{-1}{2}$$

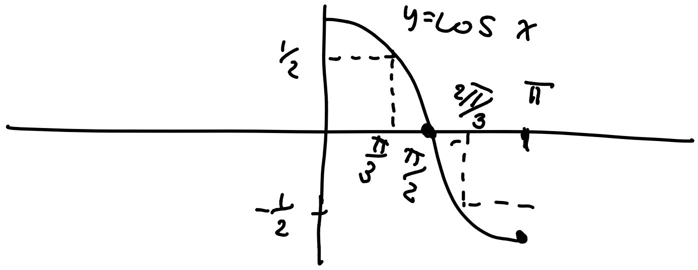
Since $0 \leq \theta \leq \pi$, this means $\theta = \frac{2\pi}{3}$.

How do we know?

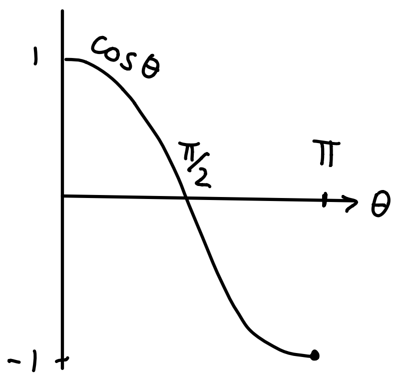
equilateral:



Set side lengths to 2,
$\alpha = \pi/3$,
Then $\cos\alpha = \frac{1}{2}$.



Notice that for $\vec{v}, \vec{w}$ nonzero vectors, $\|\vec{v}\|$ and $\|\vec{w}\|$ are both positive, so $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$ has the same sign as $\cos\theta$, so we get the following:



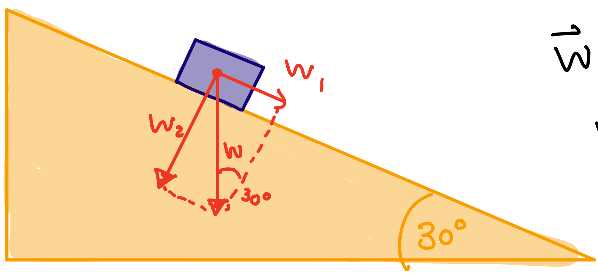
- $\vec{v} \cdot \vec{w} > 0 \iff 0 \leq \theta < \frac{\pi}{2}$, i.e. θ is acute
- $\vec{v} \cdot \vec{w} = 0 \iff \theta = \frac{\pi}{2}$, i.e. $\vec{v}$ and $\vec{w}$ are perpendicular
- $\vec{v} \cdot \vec{w} < 0 \iff \frac{\pi}{2} < \theta \leq \pi$, i.e. θ is obtuse

If $\vec{v} \cdot \vec{w} = \vec{0}$ we say that $\vec{v}$ and $\vec{w}$ are orthogonal, i.e. if the angle between them is $\pi/2$ or one of $\vec{v}$ and $\vec{w}$ is $\vec{0}$.

Projections

It's often useful to be able to write a vector as the sum of two orthogonal vectors.

Ex: A 10 lb block is on a frictionless 30° ramp. How much force is needed to keep the block from sliding?



$\vec{w}$ = force due to gravity, so $\|\vec{w}\| = 10$. We can write $\vec{w} = \vec{w}_1 + \vec{w}_2$ and find $\|\vec{w}_1\|$.

$$\frac{\|\vec{w}_1\|}{\|\vec{w}\|} = \sin 30^\circ = \frac{1}{2}.$$

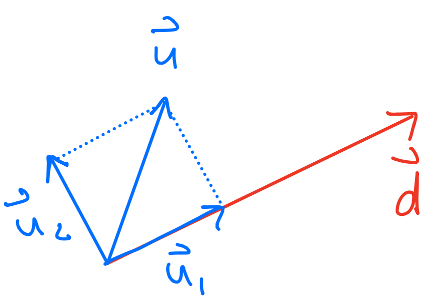
$$\Rightarrow \|\vec{w}_1\| = \frac{1}{2}\|\vec{w}\| = \frac{1}{2} 10 = 5.$$

So 5 lbs of force up the ramp are needed in order to keep the block from moving.

More generally:

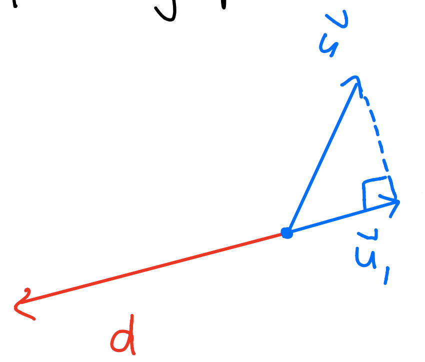
If $\vec{d}$ is a nonzero vector and $\vec{u}$ an arbitrary vector, we want to find the projection of $\vec{u}$ onto $\vec{d}$. That is, we need

$$\vec{u} = \vec{u}_1 + \vec{u}_2$$



where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{u}_1$. In this case, $\vec{u}_1$ is called the projection of $\vec{u}$ onto $\vec{d}$, denoted $u_1 = \text{proj}_{\vec{d}}\, \vec{u}$.

Note: • $\vec{u}_1$ may point the opposite direction from $\vec{d}$.



• $\vec{u}_1 = \vec{0}$ if and only if $\vec{d}$ and $\vec{u}$ are orthogonal.

How to calculate $\vec{u}_1 = \text{proj}_{\vec{d}} \vec{u}$?

We know that $\vec{u}_1$ is parallel to $\vec{d}$, so

$$\vec{u}_1 = t\vec{d}, \text{ some scalar } t.$$

We also know that $\vec{u}_2 = \vec{u} - \vec{u}_1 = \vec{u} - t\vec{d}$ is perpendicular to $\vec{d}$, so

$$0 = (\vec{u} - t\vec{d}) \cdot \vec{d} = \vec{u} \cdot \vec{d} - t\vec{d} \cdot \vec{d} = \vec{u} \cdot \vec{d} - t\|\vec{d}\|^2$$

$$\Rightarrow t\|\vec{d}\|^2 = \vec{u} \cdot \vec{d}$$

$$\Rightarrow t = \frac{\vec{u} \cdot \vec{d}}{\|\vec{d}\|^2}$$

Thus, we've shown that

$$\boxed{\text{proj}_{\vec{d}} \vec{u} = \underbrace{\left(\frac{\vec{u} \cdot \vec{d}}{\|\vec{d}\|^2}\right)}_{\text{scalar}} \vec{d}}$$

Ex: What is the projection of $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ onto $\vec{d} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$?

$\vec{u}_1 = \text{proj}_{\vec{d}}\,\vec{u} = \left(\dfrac{-1+0+8}{1+0+4}\right)\begin{bmatrix}-1\\0\\2\end{bmatrix} = \dfrac{7}{5}\begin{bmatrix}-1\\0\\2\end{bmatrix}$

Thus, $\vec{u}_2 = \vec{u} - \vec{u}_1 = \begin{bmatrix}1\\2\\4\end{bmatrix} - \dfrac{7}{5}\begin{bmatrix}-1\\0\\2\end{bmatrix}$

$$= \frac{5}{5}\begin{bmatrix}1\\2\\4\end{bmatrix} - \frac{7}{5}\begin{bmatrix}-1\\0\\2\end{bmatrix}$$

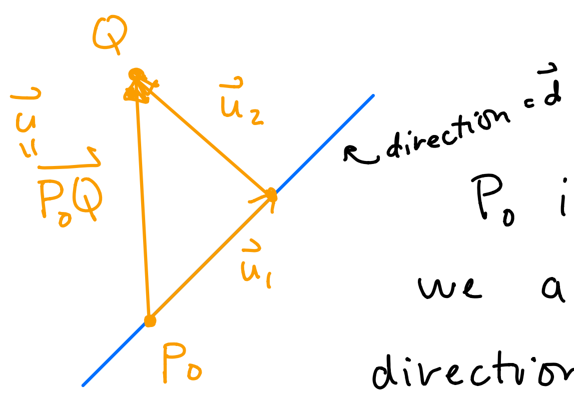
$$= \frac{1}{5}\left(\begin{bmatrix}5\\10\\20\end{bmatrix} - \begin{bmatrix}-7\\0\\14\end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix}12\\10\\6\end{bmatrix}$$

So $\vec{u} = \underbrace{\vec{u}_1}_{\text{proj. onto } \vec{d}} + \underbrace{\vec{u}_2}_{\text{perp. to } \vec{d}}$

Ex: What is the shortest distance from the point

$$Q = (1, 3, -2)$$

to the line w/ vector equation $\vec{p} = \underbrace{\begin{bmatrix}2\\0\\-1\end{bmatrix}}_{P_0} + t\underbrace{\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{\vec{d}}$?



P_0 is a point on the line, so we are looking for $\vec{u}_1$ in the direction of the line (i.e. parallel to $\vec{d}$) so that $\vec{u}_2 = \overrightarrow{P_0Q} - \vec{u}_1$ is perpendicular to $\vec{d}$. The distance we are looking for will then be $\|\vec{u}_2\|$

$$\vec{u} = \overrightarrow{P_0Q} = \begin{bmatrix} 1-2 \\ 3-0 \\ -2-(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

So $\vec{u}_1 = \text{proj}_{\vec{d}} \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} = \dfrac{\vec{u}\cdot\vec{d}}{\|\vec{d}\|^2}\vec{d} = \left(\dfrac{-1-3+0}{1+1+0}\right)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$$= \frac{-4}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

Thus, $\vec{u}_2 = \vec{u} - \vec{u}_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

So the distance is $\|\vec{u}_2\| = \sqrt{1^2+1^2+(-1)^2} = \sqrt{3}$.

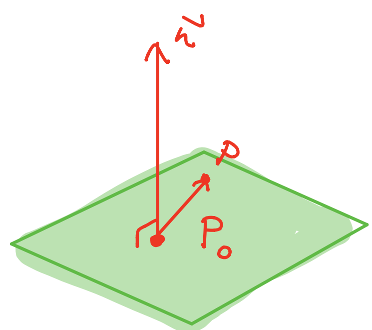
Which point on the line is closest to Q?

$$\vec{P}_0 + \vec{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}.$$

(Check that the distance from Q to $(0,2,-1)$ is $\sqrt{3}$.)

Planes

How can we describe a plane in $\mathbb{R}^3$? Note that there is a unique plane perpendicular to a given line, containing a particular point. Thus, those are the two things we will use to describe a plane.



Def: A nonzero vector is normal to a plane, if it is orthogonal to every vector in the plane.

Thus, if $P_0 = (x_0, y_0, z_0)$ is a point, and $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$, then the plane containing P_0 with normal vector $\vec{n}$ contains the point $P = (x, y, z)$ as long as

$\overrightarrow{P_0P}$ is orthogonal to $\vec{n}$.

$$\overrightarrow{P_0P} = \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}.$$

So the plane is the points $P = (x, y, z)$ that satisfy

vector equation of a plane

$$\vec{n} \cdot \overrightarrow{P_0P} = 0, \quad \text{i.e.}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$$

This gives us the following.

scalar equation of a plane:

The plane containing the point $P_0 = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \neq \vec{0}$ is given by

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

That is, a point $P = (x, y, z)$ is on the plane if and only if it satisfies this equation.

Ex: The plane with $\vec{n} = \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix}$ as a normal vector that contains the point $(5, -2, 1)$ has equation

$$2(x - 5) + 3(y + 2) - 7(z - 1) = 0.$$

Ex: Consider the plane $3x - 2y = 6$. Find a plane parallel to this plane that contains the point $(7, 0, 5)$.

All parallel planes have the same normal vectors, so we just need to find a normal vector to $3x - 2y = 6$.

The normal vector will just be the coefficients $\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$. How do we know this? We can rewrite $3x - 2y = 6$ as

$$3x - 2y - 6 = 0$$

$$\Rightarrow 3x - 2(y+3) + 0z = 0$$

coefficients

Thus, the plane we want is

$$3(x-7) - 2y + 0z = 0, \text{ or}$$

$$3(x-7) - 2y = 0.$$

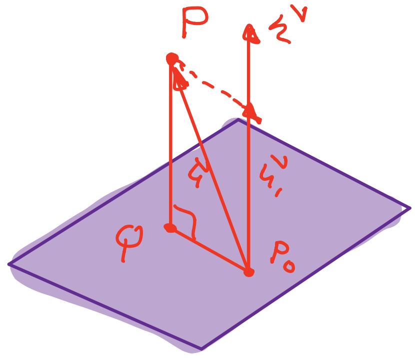
Ex: Find shortest distance from the point $P = (2, 1, -3)$ to the plane with equation $3x - y + 4z = 1$.

Rewrite equation as $3x - (y+1) + 4z = 0$

Then $P_0 = (0, -1, 0)$ is on the plane and

$\vec{n} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ is a normal vector

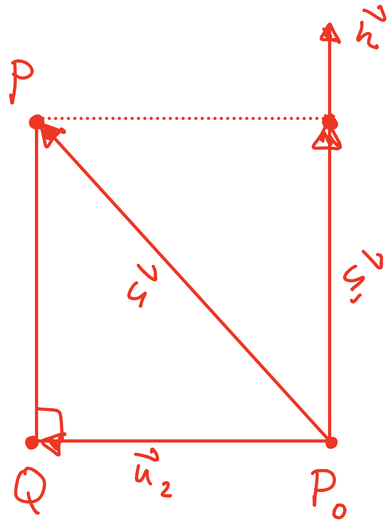
We want to find Q, the point on the plane closest to P.



Solution 1:

Let $\vec{u} = \overrightarrow{P_0P} = \begin{bmatrix} 2-0 \\ 1-(-1) \\ -3-0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$.

The projection of $\vec{u}$ onto $\vec{n}$ is:



$$\vec{u}_1 = \text{proj}_{\vec{n}} \vec{u}$$

$$= \frac{\vec{n} \cdot \vec{u}}{\|\vec{n}\|^2} \vec{n}$$

$$= \frac{6-2-12}{9+1+16} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

$$= \frac{-8}{26} \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix} = \frac{-4}{13} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

So the distance is $\|\vec{u}_1\| = \frac{4}{13} \sqrt{9+1+16}$

$$= \frac{4\sqrt{26}}{13}$$

$$\vec{u}_2 = \vec{u} - \vec{u}_1 = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \frac{4}{13} \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$$

$$= \frac{13}{13} \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix} + \frac{4}{13} \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

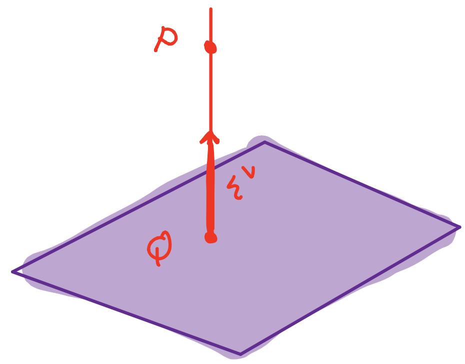
$$= \frac{1}{13} \left(\begin{bmatrix} 26 \\ 26 \\ -39 \end{bmatrix} + \begin{bmatrix} 12 \\ -4 \\ 16 \end{bmatrix} \right) = \frac{1}{13} \begin{bmatrix} 38 \\ 22 \\ -23 \end{bmatrix}$$

So to find the point we want,

$$\vec{P}_0 + \vec{u}_2 = \begin{bmatrix}0\\-1\\0\end{bmatrix} + \frac{1}{13}\begin{bmatrix}38\\22\\-23\end{bmatrix} = \frac{1}{13}\begin{bmatrix}0\\-13\\0\end{bmatrix} + \frac{1}{13}\begin{bmatrix}38\\22\\-23\end{bmatrix} = \frac{1}{13}\begin{bmatrix}38\\9\\-23\end{bmatrix}$$

so the point is $\left(\frac{38}{13}, \frac{9}{13}, \frac{-23}{13}\right)$.

<u>Solution 2:</u> We can just find a line through P perpendicular to the plane and see where it intersects the plane.



$\vec{n}$ will be a direction vector for the line, so a vector equation for the line is

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}2\\1\\-3\end{bmatrix} + t\begin{bmatrix}3\\-1\\4\end{bmatrix} \quad \text{or} \quad \begin{aligned} x &= 2+3t \\ y &= 1-t \\ z &= -3+4t \end{aligned}$$

So to find where this meets the plane, we can plug these values for x, y, z into the equation for the plane, to see if there is a t that works:

$$3x-(y+1)+4z=0$$

$$3(2+3t)-(1-t+1)+4(-3+4t)=0$$

$$6+9t-2+t-12+16t=0$$

$$\Rightarrow 26t = 8$$

$$\Rightarrow t = \frac{4}{13}$$

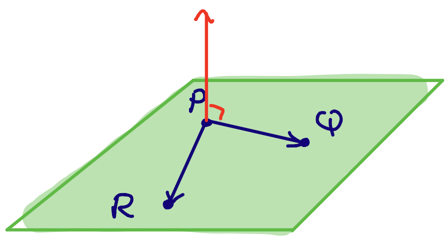
Plugging this back into the equation for the line gives us Q.

$$Q = \left(2 + \frac{12}{13},\ 1 - \frac{4}{13},\ -3 + \frac{16}{13}\right)$$
$$= \left(\frac{38}{13}, \frac{9}{13}, \frac{-23}{13}\right)$$

We can find the distance by just calculating $\|\vec{PQ}\|$, which is $\frac{4}{13}\sqrt{26}$.

The cross product

If P, Q, R are distinct points in $\mathbb{R}^3$ (not all in a line), then there is a unique plane containing all three points.



How do we find the plane?

We need to find a normal vector. i.e. a vector orthogonal to both $\vec{PR}$ and $\vec{PQ}$. The cross product gives us a way to do this.

First we give the standard basis vectors names:

$$\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

There is a trick for defining the cross product:

Def: If $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ then the <u>cross product</u> is defined

$$\vec{v}_1 \times \vec{v}_2 = \det \begin{bmatrix} \vec{i} & x_1 & x_2 \\ \vec{j} & y_1 & y_2 \\ \vec{k} & z_1 & z_2 \end{bmatrix} = \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} \vec{i} - \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} \vec{j} + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \vec{k}$$

expanded along first column

$$= (y_1 z_2 - y_2 z_1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - (x_1 z_2 - x_2 z_1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (x_1 y_2 - x_2 y_1)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} y_1 z_2 - y_2 z_1 \\ -(x_1 z_2 - x_2 z_1) \\ x_1 y_2 - x_2 y_1 \end{bmatrix}.$$

You can take either of the highlighted formulas as the definition, but the first is easier to remember.

Ex: If $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$, then

$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{i} & 1 & 0 \\ \vec{j} & 1 & 3 \\ \vec{k} & 2 & -1 \end{bmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \vec{k}$$

$$= -7\vec{i} - (-1)\vec{j} + 3\vec{k}$$

$$= \begin{bmatrix} -7 \\ 1 \\ 3 \end{bmatrix}.$$

Note that this vector is orthogonal to both $\vec{v}$ and $\vec{w}$:

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = -7 + 1 + 6 = 0, \text{ and}$$

$\vec{w} \cdot (\vec{v} \times \vec{w}) = 0 + 3 - 3 = 0.$

This is true in general:

Theorem: Let $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$.

1. $\vec{v} \times \vec{w}$ is a vector orthogonal to $\vec{v}$ and $\vec{w}$.

2. If $\vec{v}$ and $\vec{w}$ are nonzero, then $\vec{v} \times \vec{w} = \vec{0}$ if and only if $\vec{v}$ and $\vec{w}$ are parallel.

3. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$.

Ex: What is the equation of the plane through $P = (1, 3, -2)$, $Q = (1, 1, 5)$, and $R = (2, -2, 3)$?

$\overrightarrow{PQ} = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$ and $\overrightarrow{PR} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$ lie in the plane, so a normal vector to the plane is

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \vec{i} & 0 & 1 \\ \vec{j} & -2 & -5 \\ \vec{k} & 7 & 5 \end{bmatrix}$$

$$= \begin{vmatrix} -2 & -5 \\ 7 & 5 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 1 \\ 7 & 5 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 1 \\ -2 & -5 \end{vmatrix} \vec{k}$$

$$= \begin{bmatrix} 25 \\ 7 \\ 2 \end{bmatrix}$$

Thus, the plane has equation

$$25(x-1)+7(y-3)+2(z+2)=0.$$

Ex: Find the shortest distance between non parallel lines

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

First, we find a plane containing the first line, parallel to the second. Its normal vector must be orthogonal to the direction vectors for each line, so we set it to:

$$\vec{n} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \det\begin{bmatrix} \vec{i} & 2 & 1 \\ \vec{j} & 0 & 1 \\ \vec{k} & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -(-2-1) \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

Since $P_1=(1,0,-1)$ is a point on the first line, an equation for the plane is thus

$$-(x-1)+3y+2(z+1)=0$$

Now we just need to find the distance from a point on the second line, e.g. $P_2=(3,1,0)$ to the plane.

This length will just be the length of the projection of $\vec{u}=\overrightarrow{P_1P_2} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ onto $\vec{n}$.

The projection is

$$\mathrm{proj}_{\vec{n}} \vec{u} = \frac{\vec{u}\cdot\vec{n}}{\|\vec{n}\|^2}\, \vec{n},$$

which has length

$$\frac{|\vec{u}\cdot\vec{n}|}{\|\vec{n}\|^2}\|\vec{n}\| = \frac{|\vec{u}\cdot\vec{n}|}{\|\vec{n}\|} = \frac{-2+3+2}{\sqrt{1+9+4}} = \frac{3}{\sqrt{14}}$$

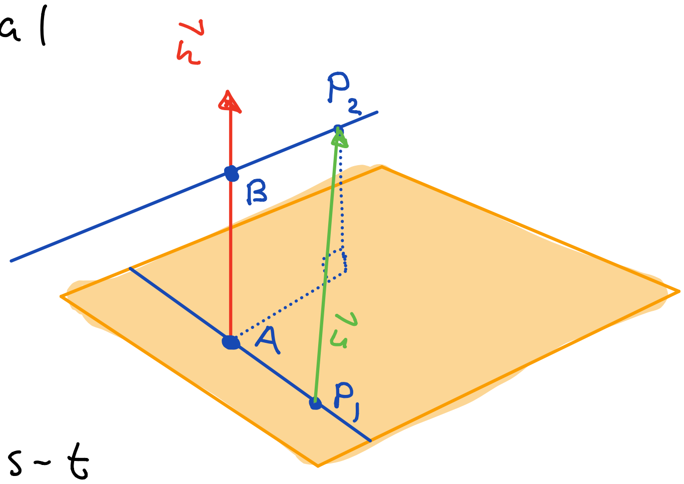
How do we find the points on the two lines where they are closest?

Say the points are A and B on the two lines, respectively. Then

$$\vec{AB} = \begin{bmatrix} 3+s-(1+2t) \\ 1+s-0 \\ -s-(-1+t) \end{bmatrix} = \begin{bmatrix} 2+s-2t \\ 1+s \\ 1-s-t \end{bmatrix}$$

for some s and t.

$\vec{AB}$ must be orthogonal to both lines, so it should be orthogonal to the respective direction vectors:



$$0 = \vec{AB}\cdot\begin{bmatrix}2\\0\\1\end{bmatrix} = 4+2s-4t+1-s-t$$
$$= 5+s-5t$$

$$0 = \vec{AB}\cdot\begin{bmatrix}1\\1\\-1\end{bmatrix} = 2+s-2t+1+s-1+s+t = 2+3s-t$$

So we get
$$5t - s = 5$$
$$t - 3s = 2$$

$$\to \left[\begin{array}{cc|c} 5 & -1 & 5 \\ 1 & -3 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 0 & 14 & -5 \\ 1 & -3 & 2 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & -5/14 \\ 1 & -3 & 2 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 0 & 1 & -5/14 \\ 1 & 0 & \frac{13}{14} \end{array}\right] \implies s = \frac{-5}{14}, \quad t = \frac{13}{14}$$

So $A = \left(\frac{40}{14}, 0, \frac{-1}{14}\right)$, $B = \left(\frac{37}{14}, \frac{9}{14}, \frac{5}{14}\right)$.

Practice problems: 4.2 : 2ab, 3, 4bd, 5, 9, 10cd, 11bc, 12a, 13b, 14adeh, 15bc, 16b, 19a, 23a, 24a